

PREDUALS OF QUADRATIC CAMPANATO SPACES ASSOCIATED TO OPERATORS WITH HEAT KERNEL BOUNDS

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ABSTRACT. Let L be a nonnegative, self-adjoint operator on $L^2(\mathbb{R}^n)$ with the Gaussian upper bound on its heat kernel. As a generalization of the square Campanato space $\mathcal{L}_{-\Delta}^{2,\lambda}(\mathbb{R}^n)$, in [16] the quadratic Campanato space $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ is defined by a variant of the maximal function associated with the semigroup $\{e^{-tL}\}_{t \geq 0}$. On the basis of [14] and [35] this paper addresses the preduality of $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ through an induced atom (or molecular) decomposition. Even in the case $L = -\Delta$ the discovered predual result is new and natural.

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1. INTRODUCTION

Given $1 \leq p < \infty$ and $0 < \lambda < n$. A locally integrable complex-valued function f on \mathbb{R}^n is said to belong to be in the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ provided

$$\|f\|_{L^{p,\lambda}} = \sup_{B \subset \mathbb{R}^n} \left(r_B^{-\lambda} \int_B |f(x)|^p dx \right)^{1/p} < \infty,$$

where r_B is the radius of the ball B . Such a function space was introduced by C. B. Morrey in [27] to treat the solutions of some quasi-linear elliptic PDEs. Since then, the theory of Morrey spaces has been developed extensively; see e.g. [3, 4, 7, 29, 24, 32] and the references therein.

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To weaken the integral condition appeared in the Morrey space, in his 1963/4 papers [8, 9] S. Campanato utilized the modified mean oscillation to define the following function space:

$$f \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n) \iff \|f\|_{\mathcal{L}^{p,\lambda}} = \sup_{B \subset \mathbb{R}^n} \left(r_B^{-\lambda} \int_B |f(x) - f_B|^p dx \right)^{1/p} < \infty,$$

where $f_B := |B|^{-1} \int_B f(y) dy$. It is easy to see that $L^{p,\lambda}(\mathbb{R}^n)$ is a proper subclass of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ since any complex constant is in $\mathcal{L}^{p,\lambda}(\mathbb{R}^n) \setminus L^{p,\lambda}(\mathbb{R}^n)$. Interestingly, $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ under $(p, \lambda) \in (1, \infty) \times (0, n)$ exists as a dual space - more precisely, if $Z^{q,\lambda}(\mathbb{R}^n)$ with $q = p/(p-1)$ stands for the Zorko space (cf.[37]) of all functions f on \mathbb{R}^n with the norm

$$\|f\|_{Z^{q,\lambda}} = \inf \left\{ \|\{c_k\}\|_{l^1} : f = \sum_k c_k a_k \right\} < +\infty,$$

where a_k is a (q, λ) -atom and $\|\{c_k\}\|_{l^1} < +\infty$, and the infimum is taken over all possible functions $f = \sum_k c_k a_k$ whose a_k is a (q, λ) -atom on \mathbb{R}^n :

- a_k is supported on a ball $B \subseteq \mathbb{R}^n$;
- $\int a(x) dx = 0$;
- $\|a\|_q \leq r_B^{-\lambda/p}$ with $1/q + 1/p = 1$,

then

$$(Z^{q,\lambda}(\mathbb{R}^n))^* = \mathcal{L}^{p,\lambda}(\mathbb{R}^n),$$

namely, the Zorko $Z^{q,\lambda}(\mathbb{R}^n)$ is identified with a predual of the Campanato space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$.

However, there are important situations in which the standard theory of function spaces is not applicable, including certain problems in the theory of partial differential operators generalizing the Laplacian. There is a need to consider the function spaces that are adapted to a linear operator L , similarly to the way that such function spaces as the above-defined Campanato spaces are adapted to the Laplacian. This topic has attracted a lot of attention, and has been a very active research topic in harmonic analysis, potential theory and PDEs; see, for instance, [19, 5, 17, 18, 6, 21, 22, 20, 15, 23, 31].

For our purpose, we will consider such a nonnegative self-adjoint operator L on $L^2(\mathbb{R}^n)$ that the semigroup e^{-tL} , generated by $-L$ on $L^2(\mathbb{R}^n)$, has the kernel $p_t(x, y)$ obeying the Gaussian upper bound for a constant $C > 0$:

$$(1.1) \quad |p_t(x, y)| \leq C t^{-n/2} \exp \left(-\frac{|x-y|^2}{ct} \right) \quad \text{for all } t > 0 \text{ \& for a.e. } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Such an upper bound condition is a typical one needed in the theory of elliptic or sub-elliptic differential operators of second order; see, for example, [12].

Keeping in mind that the quadratic Campanato space $\mathcal{L}^{2,\lambda}(\mathbb{R}^n)$ is a prime example in the family of the Campanato spaces that are useful in analysis and PDEs (see e.g. [33, 34] and their references), and following [16], we say that a function f belongs to the space $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ provided

$$\|f\|_{\mathcal{L}_L^{2,\lambda}} = \sup_{B \subset \mathbb{R}^n} \left(r_B^{-\lambda} \int_B |f(x) - e^{-r_B^2 L} f(x)|^2 dx \right)^{1/2} < \infty,$$

where r_B is the radius of the ball B . Here, the function $e^{-r_B^2 L} f$ is seen as an average version of f (at the scale r_B^2) and replaces the mean value f_B in the definition of the Campanato space $\mathcal{L}^{2,\lambda}(\mathbb{R}^n)$. For this idea and its applications, we refer the reader to [13, 25, 17, 18, 16], and especially point out that if L equals the nonnegative Laplace operator $-\Delta = -\sum_{j=1}^n \partial^2/\partial x_j^2$ on \mathbb{R}^n , then $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ coincides with $\mathcal{L}^{2,\lambda}(\mathbb{R}^n)$, see [16, Proposition 8]. Hence, $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ generalizes $\mathcal{L}^{2,\lambda}(\mathbb{R}^n)$.

Needless to say, the study of $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ is far beyond completeness. Nevertheless, being inspired by the Choquet integral against the Hausdorff capacity used in [14] and [35], in this paper we can at least obtain the following description of the preduality for $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ as one of the fundamental problems of the Campanato function theory associated to a nonnegative self-adjoint operator with the Gaussian kernel bound (1.1).

Theorem 1.1. *Let L be a nonnegative self-adjoint operator obeying (1.1) and $\Lambda_\lambda^{(\infty)}$ be $(0, n) \ni \lambda$ -dimensional Hausdorff capacity. If $F\dot{H}_L^\lambda(\mathbb{R}^n)$ stands for the completion of*

$$\left\{ f \in L^2(\mathbb{R}^n) : \|f\|_{F\dot{H}_L^\lambda} = \inf_{\omega} \left(\int_{\mathbb{R}_+^{n+1}} |t^2 L e^{-t^2 L} f(x)|^2 \omega(x, t)^{-1} \frac{dx dt}{t} \right)^{1/2} < \infty \right\}$$

in the norm $\|\cdot\|_{F\dot{H}_L^\lambda}$, where the infimum is taken over all nonnegative Borel functions ω on \mathbb{R}_+^{n+1} with its non-tangential maximal function $\mathbf{N}\omega$ satisfying the Choquet integral condition $\int_{\mathbb{R}^n} \mathbf{N}\omega d\Lambda_\lambda^{(\infty)} \leq 1$, then the predual of the space $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ is $F\dot{H}_L^\lambda(\mathbb{R}^n)$, namely,

$$(F\dot{H}_L^\lambda(\mathbb{R}^n))^* = \mathcal{L}_L^{2,\lambda}(\mathbb{R}^n).$$

The proof of the above preduality theorem proceeds via the forthcoming three sections. In Section 2, we recall some basic facts about Choquet integrals and square tent spaces. In Section 3, we make the space $F\dot{H}_L^\lambda(\mathbb{R}^n)$ more transparent via giving its atomic (or molecular) decomposition. In Section 4, we use a new description of $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ and the atomic (or molecular) characterization of $F\dot{H}_L^\lambda(\mathbb{R}^n)$ as a tool to complete the argument for Theorem 1.1.

Remark 1.2. (i) *Hopefully, the investigation of $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ and hence $F\dot{H}_L^\lambda(\mathbb{R}^n)$ can be moved appropriately to a more general setting of operators on metric spaces as described in [28, Chapter 7].*

(ii) *From now on, the letters C and c will denote (possibly different) constants that are independent of the essential variables.*

2. NECESSARY GROUNDWORK

2.1. Choquet integrals. We shall work exclusively with the upper half-space \mathbb{R}_+^{n+1} . If $x \in \mathbb{R}^n$, $\Gamma(x)$ will denote the cone $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$. For any set $E \subset \mathbb{R}^n$, the tent over E , $T(E)$, is the set $\{(y, t) \in \mathbb{R}_+^{n+1} : B(y, t) \subset E\}$. The nontangential maximal function $\mathbf{N}f$ of a measurable function f on \mathbb{R}_+^{n+1} is defined by

$$\mathbf{N}f(x) = \sup_{(y,t) \in \Gamma(x)} |f(y, t)|.$$

Let us recall the notion of Hausdorff capacities; see, for example, [1, 2].

Definition 2.1. *If $\lambda \in (0, n)$ and $E \subset \mathbb{R}^n$, then λ -dimensional Hausdorff capacity of E is defined by*

$$\Lambda_\lambda^{(\infty)}(E) := \inf \left\{ \sum_j r_j^\lambda : E \subset \bigcup_{j=1}^{\infty} B(x_j, r_j) \right\},$$

where the infimum is taken over all covers of E by balls $B(x_j, r_j)$ with centers x_j and radii r_j .

A dyadic version of the Hausdorff capacity, $\tilde{\Lambda}_\lambda^{(\infty)}$, was introduced in [36], which is defined by

$$\tilde{\Lambda}_\lambda^{(\infty)}(E) = \inf \left\{ \sum_j l(I_j)^\lambda : E \subset \left(\bigcup_j I_j \right)^\circ \right\},$$

where the infimum ranges only over covers of E by dyadic cubes $\{I_j\}_j$, and A° denotes the interior of the set A .

It is well known that λ -dimensional Hausdorff capacity $\Lambda_\lambda^{(\infty)}$ and $\tilde{\Lambda}_\lambda^{(\infty)}$ are equivalent – more precisely, there exist positive constants $C_1(n, \lambda)$ and $C_2(n, \lambda)$, depending on n and λ , such that

$$(2.1) \quad C_1(n, \lambda) \Lambda_\lambda^{(\infty)}(E) \leq \tilde{\Lambda}_\lambda^{(\infty)}(E) \leq C_2(n, \lambda) \Lambda_\lambda^{(\infty)}(E), \quad \text{for all } E \subset \mathbb{R}^n.$$

Next, we recall a notion of the Choquet integrals with respect to the Hausdorff capacities (cf. [1, 2]): for a function $f : \mathbb{R}^n \rightarrow [0, \infty]$, define

$$\int_{\mathbb{R}^n} f d\Lambda_\lambda^{(\infty)} := \int_0^\infty \Lambda_\lambda^{(\infty)}(\{x \in \mathbb{R}^n : f(x) > t\}) dt.$$

2.2. Square tent spaces. Definitions 2.2 & 2.4 below are inspired by [14].

Definition 2.2. *Let $\lambda \in (0, n)$. The space $F\dot{T}^\lambda(\mathbb{R}_+^{n+1})$ consists of all Lebesgue measurable functions f on \mathbb{R}_+^{n+1} for which*

$$\|f\|_{F\dot{T}^\lambda} = \inf_{\omega} \left(\int_{\mathbb{R}_+^{n+1}} \frac{|f(x, t)|^2}{\omega(x, t)} \frac{dx dt}{t} \right)^{1/2} < \infty,$$

where the infimum is taken over all nonnegative Borel functions ω on \mathbb{R}_+^{n+1} with

$$(2.2) \quad \int_{\mathbb{R}^n} N\omega d\Lambda_\lambda^{(\infty)} \leq 1,$$

and with the restriction that ω is allowed to vanish only where f vanishes.

Note that if a function ω satisfies (2.2), then $\omega(x, t) \leq Ct^{-\lambda}$. This shows that condition $\|f\|_{F\dot{T}^\lambda} = 0$ implies $f = 0$ almost everywhere (see [14, 35]).

The following lemma shows that $\|f\|_{F\dot{T}^\lambda}$ satisfies the triangle inequality with a constant, and then $\|\cdot\|_{F\dot{T}^\lambda}$ is a quasi-norm. It can be shown that the space $F\dot{T}^\lambda(\mathbb{R}_+^{n+1})$ is complete under this quasi-norm.

Lemma 2.3. *Let $\lambda \in (0, n)$. If $\sum_j \|g_j\|_{F\dot{T}^\lambda} < \infty$, then $g = \sum_j g_j \in F\dot{T}^\lambda(\mathbb{R}_+^{n+1})$ with*

$$\|g\|_{F\dot{T}^\lambda} \leq \sqrt{C_1(n, \lambda)^{-1} C_2(n, \lambda)} \sum_j \|g_j\|_{F\dot{T}^\lambda},$$

where $C_1(n, \lambda)$ and $C_2(n, \lambda)$ are the constants in (2.1).

Proof. The proof follows from a slight modification of an argument as in [14, Lemma 5.3]. We omit the detail here. \square

Definition 2.4. *Let $\lambda \in (0, n)$. A function a on \mathbb{R}_+^{n+1} is said to be an $F\dot{T}^\lambda$ -atom associated with a ball B , if a is supported in $T(B)$ and satisfies*

$$\int_{T(B)} |a(x, t)|^2 \frac{dx dt}{t} \leq r_B^{-\lambda}.$$

Recall that the space $T_2^2(\mathbb{R}_+^{n+1})$ is a tent space (see [10]), which is defined by

$$T_2^2(\mathbb{R}_+^{n+1}) = \left\{ f(y, t) : x \mapsto \left(\int_0^\infty \int_{|y-x|<t} |f(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \text{ is in } L^2(\mathbb{R}^n) \right\}.$$

Theorem 2.5. *Let $\lambda \in (0, n)$. Then the following results hold:*

(i) $f \in F\dot{T}^\lambda(\mathbb{R}_+^{n+1})$ if and only if there is a sequence $\{a_j\}$ of $F\dot{T}^\lambda$ -atoms and an l^1 -sequence $\{\lambda_j\}$ such that

$$(2.3) \quad f = \sum_j \lambda_j a_j.$$

Moreover,

$$\|f\|_{F\dot{T}^\lambda} \approx \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\},$$

where the infimum is taken over all possible forms f in (2.3). The right hand side thus defines a norm on $F\dot{T}^\lambda(\mathbb{R}_+^{n+1})$ which makes it into a Banach spaces.

(ii) If $f \in F\dot{T}^\lambda(\mathbb{R}_+^{n+1}) \cap T_2^2(\mathbb{R}_+^{n+1})$, then the decomposition (2.3) also converges in $T_2^2(\mathbb{R}_+^{n+1})$.

Proof. The proof of (i) is similar to that of [35, Theorem 4.1] or [14, Theorem 5.4]. For the proof of (ii), we can follow an argument of [20, Proposition 4.10]) to show it, and so omit details here. \square

3. THE SPACE $F\dot{H}_L^\lambda(\mathbb{R}^n)$

3.1. An atomic decomposition of $F\dot{H}_L^\lambda(\mathbb{R}^n)$. Given a nonnegative self-adjoint operator L on $L^2(\mathbb{R}^n)$ satisfying (1.1). For any $(x, t) \in \mathbb{R}^n \times (0, +\infty) = \mathbb{R}_+^{n+1}$ and for every $f \in L^2(\mathbb{R}^n)$, define

$$\begin{cases} P_t f(x) = e^{-tL} f(x) = \int_{\mathbb{R}^n} p_t(x, y) f(y) dy; \\ Q_t f(x) = t L e^{-tL} f(x) = \int_{\mathbb{R}^n} -t \left(\frac{dp_t(x, y)}{dt} \right) f(y) dy. \end{cases}$$

Then, like $p_t(x, y)$ obeying (1.1) the kernel $q_t(x, y)$ of Q_t satisfies

$$(3.1) \quad |q_t(x, y)| \leq Ct^{-n/2} \exp\left(-\frac{|x-y|^2}{ct}\right) \text{ for all } t > 0 \text{ \& for a.e. } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

See, for instance, [28, Theorem 6.17].

Definition 3.1. Let $\lambda \in (0, n)$ and L be a nonnegative self-adjoint operator obeying (1.1). The space $F\dot{H}_L^\lambda(\mathbb{R}^n)$ is defined to be the completion of

$$\left\{ f \in L^2(\mathbb{R}^n) : \|f\|_{F\dot{H}_L^\lambda} = \|t^2 L e^{-t^2 L}(f)\|_{F\dot{T}^\lambda} < \infty \right\}$$

in the norm $\|\cdot\|_{F\dot{H}_L^\lambda}$.

Definition 3.2. Given $(M, \lambda) \in \mathbb{N} \times (0, n)$ and L , a nonnegative self-adjoint operator enjoying (1.1).

(i) A function $a \in L^2(\mathbb{R}^n)$ is called a $(2, M, \lambda)$ -atom associated to the operator L if there exist a function $b \in \mathcal{D}(L^M)$ and a ball B such that

- $a = L^M b$;
- $\text{supp } L^k b \subset B$, $k = 0, 1, \dots, M$;
- $\|(r_B^2 L)^k b\|_{L^2(\mathbb{R}^n)} \leq r_B^{2M-\lambda/2}$, $k = 0, 1, \dots, M$.

(ii) We say that $\sum \lambda_j a_j$ is an atomic $(2, M, \lambda)$ -representation of f if $\{\lambda_j\}_{j=0}^\infty \in \ell^1$, each a_j is a $(2, M, \lambda)$ -atom, and the sum converges in $L^2(\mathbb{R}^n)$. Set

$$\mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n) := \left\{ f : f \text{ has an atomic } (2, M, \lambda)\text{-representation} \right\},$$

with the norm given by

$$\begin{aligned} & \|f\|_{\mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)} \\ &= \inf \left\{ \sum_{j=0}^\infty |\lambda_j| : f = \sum_{j=0}^\infty \lambda_j a_j \text{ is an atomic } (2, M, \lambda)\text{-representation} \right\}. \end{aligned}$$

(iii) The space $F\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)$ is then defined as the completion of $\mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)$ with respect to $\|\cdot\|_{\mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)}$.

Recall that, if $E_L(\lambda)$ denotes the spectral decomposition of a nonnegative self-adjoint operator L on $L^2(\mathbb{R}^n)$, then for every bounded Borel function $F : [0, \infty) \rightarrow \mathbb{C}$, one defines the operator $F(L) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by the formula

$$(3.2) \quad F(L) := \int_0^\infty F(\lambda) dE_L(\lambda).$$

Hence the operator $\cos(t\sqrt{L})$ is well-defined on $L^2(\mathbb{R}^n)$ for all $t > 0$. Thus it makes sense to make the following definition.

Definition 3.3. A nonnegative self-adjoint operator L is said to satisfy the finite speed propagation property for solutions of the corresponding wave equation if there exists a constant $c_0 > 0$ such that

$$(3.3) \quad \langle \cos(t\sqrt{L})f_1, f_2 \rangle = \int \cos(t\sqrt{L})f_1(x) \overline{f_2(x)} dx = 0$$

for all $0 < c_0 t < d(U_1, U_2)$ and $U_i \subset \mathbb{R}^n$, $f_i \in L^2(U_i)$, $i = 1, 2$.

In particular, if $K_{\cos(t\sqrt{L})}(x, y)$ denotes the integral kernel of the operator $\cos(t\sqrt{L})$, then (3.3) entails that for every $t > 0$,

$$(3.4) \quad \text{supp } K_{\cos(t\sqrt{L})} \subseteq \mathcal{D}_t := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq c_0 t\}.$$

Proposition 3.4. *Let L be a nonnegative self-adjoint operator acting on $L^2(\mathbb{R}^n)$. Then (1.1) implies (3.3).*

Proof. The argument follows from [30, Theorem 2] and [11, Theorem 3.4]. \square

From Proposition 3.4 and (1.1) it follows that the kernel $K_{\cos(t\sqrt{L})}(x, y)$ of the operator $\cos(t\sqrt{L})$ has the property (3.4). By the Fourier inversion formula, whenever F is an even bounded Borel function with $\hat{F} \in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. Concretely, by recalling (3.2) we have

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t\sqrt{L}) dt,$$

which, when combined with (3.4), gives

$$K_{F(\sqrt{L})}(x, y) = (2\pi)^{-1} \int_{|t| \geq c_0^{-1} d(x, y)} \hat{F}(t) K_{\cos(t\sqrt{L})}(x, y) dt.$$

Lemma 3.5. *Assume L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying (1.1). Let $\varphi \in C_0^\infty(\mathbb{R})$ be even, $\text{supp } \varphi \subset [-c_0^{-1}, c_0^{-1}]$, where c_0 is the constant in (3.4). Let Φ denote the Fourier transform of φ . Then for each $\kappa = 0, 1, \dots$, and for every $t > 0$, the kernel $K_{(t^2 L)^\kappa \Phi(t\sqrt{L})}(x, y)$ of $(t^2 L)^\kappa \Phi(t\sqrt{L})$ satisfies*

$$\text{supp } K_{(t^2 L)^\kappa \Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}.$$

Proof. This follows from [20, Lemma 3.5]. \square

In what follows, let φ , c_0 , and Φ be as in Lemma 3.5, but with an extra assumption that

$$\varphi \geq 0 \quad \& \quad \varphi \geq c > 0 \quad \text{on} \quad (-1/(2c_0), 1/(2c_0)).$$

For $M \in \mathbb{N}$ set

$$\Psi(x) := x^{2(M+1)} \Phi(x), \quad \forall x \in \mathbb{R}.$$

Consider the operator $\pi_{\Psi, L} : T_2^2(\mathbb{R}_+^{n+1}) \rightarrow L^2(\mathbb{R}^n)$, given by

$$\pi_{\Psi, L}(F)(x) := \int_0^\infty \Psi(t\sqrt{L})(F(\cdot, t))(x) \frac{dt}{t},$$

where the improper integral converges weakly in L^2 . The bound

$$(3.5) \quad \|\pi_{\Psi, L} F\|_{L^2} \leq C \|F\|_{T_2^2}$$

follows readily by duality and the L^2 quadratic estimate. Moreover, we have the following analogue of the well-known argument of [10, Theorem 6].

Lemma 3.6. *Given a nonnegative self-adjoint operator L obeying (1.1). Suppose A is an $F\dot{T}^\lambda$ -atom associated to a ball B . Then there is a constant C , depending only on Ψ , such that $C^{-1}\pi_{\Psi,L}(A)$ is a $(2, M, \lambda)$ -atom associated to $2B$.*

Proof. Fix a ball B and let A be an $F\dot{T}^\lambda$ -atom associated to B . Thus,

$$\int_{T(B)} |A(x, t)|^2 \frac{dxdt}{t} \leq |B|^{-\lambda/n}.$$

For $M \in \mathbb{N}$ write

$$a := \pi_{\Psi,L}(A) = L^M b,$$

where

$$b := \int_0^\infty t^{2M} t^2 L \Phi(-t\sqrt{L})(A(\cdot, t)) \frac{dt}{t}.$$

Observe that the functions $L^k b$, $k = 0, 1, \dots, M$, are supported on the ball $2B$, by Lemma 3.5, since A is supported in $T(B)$. Consider some $g \in L^2(2B)$ such that $\|g\|_{L^2(2B)} = 1$. Then for every $k = 0, 1, \dots, M$ we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} (r_B^2 L)^k b(x) g(x) dx \right| \\ &= \left| \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^n} \left(\int_\delta^{1/\delta} t^{2M} r_B^{2k} L^k t^2 L \Phi(t\sqrt{L})(A(\cdot, t))(x) \frac{dt}{t} \right) g(x) dx \right| \\ &= \left| \int_{T(B)} A(x, t) t^{2M} r_B^{2k} L^k t^2 L \Phi(t\sqrt{L}) g(x) \frac{dxdt}{t} \right| \\ &\leq r_B^{2M} \left(\int_{T(B)} |A(x, t)|^2 \frac{dxdt}{t} \right)^{1/2} \left(\int_{T(B)} |(t^2 L)^{k+1} \Phi(t\sqrt{L}) g(x)|^2 \frac{dxdt}{t} \right)^{1/2} \\ &\leq C r_B^{2M} |B|^{-\lambda/2n} \|g\|_{L^2(2B)}, \end{aligned}$$

where the fact that A is an $F\dot{T}^\lambda$ -atom supported in $T(B)$ (hence, $0 < t < r_B$) has been used, and the last inequality follows from the L^2 quadratic estimate:

$$\left(\int_0^\infty \int_{\mathbb{R}^n} |(t^2 L)^{k+1} \Phi(t\sqrt{L}) g(x)|^2 \frac{dxdt}{t} \right)^{1/2} \leq C \|g\|_{L^2(\mathbb{R}^n)}.$$

As a consequence, one gets

$$\|(r_B^2 L)^k b\|_{L^2(2B)} \leq C r_B^{2M} |B|^{-\lambda/2n}, \quad k = 0, 1, \dots, M.$$

The proof is complete. \square

Theorem 3.7. *Assume L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying (1.1). Then*

$$F\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n) = F\dot{H}_L^\lambda(\mathbb{R}^n).$$

Moreover

$$\|f\|_{F\dot{H}_{L,at,M}^\lambda} \approx \|f\|_{F\dot{H}_L^\lambda},$$

where the implicit constants depend only on the pair $(M, \lambda) \in \mathbb{N} \times (0, n)$ and the constant in (1.1).

Proof. On the one hand, we show

$$\dot{\mathbb{F}}\dot{\mathbb{H}}_{L,at,M}^\lambda(\mathbb{R}^n) \subseteq (L^2(\mathbb{R}^n) \cap F\dot{H}_L^\lambda(\mathbb{R}^n)).$$

Note that $\dot{\mathbb{F}}\dot{\mathbb{H}}_{L,at,M}^\lambda(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$. Indeed, by definition, a $(2, M, \lambda)$ -atom belongs to $R(L)$, and therefore so does any finite linear combination of atoms. Moreover, every $f \in \dot{\mathbb{F}}\dot{\mathbb{H}}_{L,at,M}^\lambda(\mathbb{R}^n)$ is an L^2 limit of such a finite linear combination. Meanwhile, we are required to verify $\dot{\mathbb{F}}\dot{\mathbb{H}}_{L,at,M}^\lambda(\mathbb{R}^n) \subseteq F\dot{H}_L^\lambda$. To do so, let $f \in \dot{\mathbb{F}}\dot{\mathbb{H}}_{L,at,M}^\lambda(\mathbb{R}^n)$, where $f = \sum \lambda_j a_j$ is an atomic $(2, M, \lambda)$ -representation such that

$$\|f\|_{\dot{\mathbb{F}}\dot{\mathbb{H}}_{L,at}^\lambda(\mathbb{R}^n)} \approx \sum_{j=0}^{\infty} |\lambda_j|.$$

Since the sum converges in L^2 (by Definition 3.2), and an application of the L^2 boundedness of Q_{t^2} , we have that

$$|Q_{t^2}(f)| \leq \sum_{j=0}^{\infty} |\lambda_j| |Q_{t^2}(a_j)|.$$

By Lemma 2.3, it will be enough to show that for every $(2, M, \lambda)$ -atom a associated to a ball $B = B(x_B, r_B)$, we have $\|Q_{t^2}(a)\|_{F\dot{T}^\lambda} \leq C$. Now, fix $\delta = (n - \lambda)/2 > 0$ and let

$$(3.6) \quad \tilde{\omega}(x, t) = kr_B^{-\lambda} \min \left\{ 1, \left(\frac{r_B}{\sqrt{|x - x_B|^2 + t^2}} \right)^{\lambda + \delta} \right\},$$

where k will be chosen below. Since for $x \in \mathbb{R}^n$, the distance in \mathbb{R}_+^{n+1} from the $\Gamma(x)$ to $(x_B, 0)$ is $|x - x_B|/\sqrt{2}$, the nontangential maximal function of $\tilde{\omega}$ is bounded by

$$N\tilde{\omega}(x) \leq kr_B^{-\lambda} \min \left\{ 1, \left(\frac{\sqrt{2}r_B}{|x - x_B|} \right)^{\lambda + \delta} \right\}$$

and so

$$\begin{aligned} k^{-1} \int_{\mathbb{R}^n} N\tilde{\omega} d\Lambda_\lambda^{(\infty)} &\leq \int_{\mathbb{R}^n} r_B^{-\lambda} \min \left\{ 1, \left(\frac{\sqrt{2}r_B}{|x - x_B|} \right)^{\lambda + \delta} \right\} d\Lambda_\lambda^{(\infty)} \\ &= \int_0^\infty \Lambda_\lambda^{(\infty)} \{x \in \mathbb{R}^n : r_B^{-\lambda} \min \left\{ 1, \left(\frac{\sqrt{2}r_B}{|x - x_B|} \right)^{\lambda + \delta} \right\} > \alpha\} d\alpha \\ &\leq \int_0^{r_B^{-\lambda}} \Lambda_\lambda^{(\infty)} \{B(x_B, (\alpha r_B^\lambda)^{\frac{1}{\lambda + \delta}} \sqrt{2}r_B)\} d\alpha = C. \end{aligned}$$

Upon choosing $k = C^{-1}$ to make $\tilde{\omega}$ satisfy (2.2), we see

$$\begin{aligned} \|Q_{t^2}(a)\|_{F\dot{T}^\lambda}^2 &\leq \int_{\mathbb{R}_+^{n+1}} \frac{|Q_{t^2}a(x)|^2}{\tilde{\omega}(x, t)} \frac{dxdt}{t} \\ &\leq \int_{T(2B)} \frac{|Q_{t^2}a(x)|^2}{\tilde{\omega}(x, t)} \frac{dxdt}{t} + \sum_{j=2}^{\infty} \int_{T(2^j B) \setminus T(2^{j-1} B)} \frac{|Q_{t^2}a(x)|^2}{\tilde{\omega}(x, t)} \frac{dxdt}{t} \\ &=: A_0 + \sum_{j=2}^{\infty} A_j. \end{aligned}$$

Note that

$$\tilde{\omega}(x, t) \geq k2^{-(\lambda+\delta)} r_B^{-\lambda} \quad \text{on } T(2B).$$

So, by using (3.1) and the definition of $(2, M, \lambda)$ -atom we obtain

$$A_0 \leq Cr_B^\lambda \int_{\mathbb{R}_+^{n+1}} |Q_{t^2} a(x)|^2 \frac{dxdt}{t} \leq Cr_B^\lambda \|a\|_{L^2}^2 \leq C.$$

To estimate A_j for $j = 2, 3, \dots$, notice that for each $(x, t) \in T(2^j B) \setminus T(2^{j-1} B)$ one has

$$\tilde{\omega}(x, t) = Cr_B^{-\lambda} \left(\frac{r_B}{\sqrt{|x - x_B|^2 + t^2}} \right)^{\lambda+\delta} \geq Cr_B^{-\lambda} 2^{-j(\lambda+\delta)}.$$

Thus

$$(3.7) \quad A_j \leq Cr_B^\lambda 2^{j(\lambda+\delta)} \int_{T(2^j B) \setminus T(2^{j-1} B)} |Q_{t^2} a(x)|^2 \frac{dxdt}{t}.$$

For $(x, t) \in T(2^j B) \setminus T(2^{j-1} B)$ and $y \in B$, one has $t + |x - y| \geq 2^{j-1} r_B$. By (3.1), we get

$$\begin{aligned} & \int_{T(2^j B) \setminus T(2^{j-1} B)} |Q_{t^2} a(x)|^2 \frac{dxdt}{t} \\ & \leq C \int_{T(2^j B) \setminus T(2^{j-1} B)} \left(\int \frac{t |a(y)|}{(t + |x - y|)^{n+1}} dy \right)^2 \frac{dxdt}{t} \\ & \leq C \frac{1}{(2^j r_B)^{2n+2}} \|a\|_{L^1}^2 \int_{T(2^j B) \setminus T(2^{j-1} B)} t \, dxdt \\ & \leq C \frac{1}{(2^j r_B)^{2n+2}} \|a\|_{L^2}^2 |B| (2^j r_B)^2 |2^j B| \\ & \leq C 2^{-jn} r_B^{-\lambda}, \end{aligned}$$

which, combining with (3.7), implies

$$A_j \leq C 2^{-j(n-\lambda-\delta)} = C 2^{-j(n-\lambda)/2},$$

thereby deriving $\|a\|_{F\dot{H}_L^\lambda} \leq C$.

On the other hand, we verify the reverse inequality

$$F\dot{H}_L^\lambda(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \subseteq \mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n).$$

Let

$$f \in F\dot{H}_L^\lambda(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \quad \& \quad F(\cdot, t) := t^2 L e^{-t^2 L} f(\cdot).$$

Note that $F \in F\dot{T}^\lambda \cap T_2^2$ follows from the definition of $F\dot{H}_L^\lambda$. So, by Theorem 2.5 we have

$$F = \sum_j \lambda_j A_j,$$

where each A_j is a $F\dot{T}^\lambda$ -atom, the sum converges in both $T_2^2(\mathbb{R}_+^{n+1})$ and $F\dot{T}^\lambda(\mathbb{R}_+^{n+1})$, and

$$(3.8) \quad \sum_j |\lambda_j| \leq C \|F\|_{F\dot{T}} = C \|f\|_{F\dot{H}_L^\lambda}.$$

Also, by L^2 -functional calculus ([26]), we have the ‘‘Calderón reproducing formula’’

$$(3.9) \quad f(x) = c_\Psi \int_0^\infty \Psi(t\sqrt{L})(t^2 L e^{-t^2 L} f)(x) \frac{dt}{t} = c_\Psi \pi_{\Psi,L}(F) = c_\Psi \sum_j \lambda_j \pi_{\Psi,L}(A_j).$$

where the last sum converges in $L^2(\mathbb{R}^n)$ by (3.5). Moreover, by Lemma 3.6 we have that up to multiplication by some harmless constant C , each $a_j := c_\Psi \pi_{\Psi,L}(A_j)$ is a $(2, M, \lambda)$ -atom. Consequently, the last sum in (3.9) is an atomic $(2, M, \lambda)$ -representation, so that $f \in \mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)$, and by (3.8) we have

$$\|f\|_{\mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)} \leq C \|f\|_{F\dot{H}_L^\lambda},$$

whence deriving the desired inclusion.

The above argument shows that both $F\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)$ and $F\dot{H}_L^\lambda(\mathbb{R}^n)$ have the same dense subset $\mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap F\dot{H}_L^\lambda(\mathbb{R}^n)$ with equivalent norms, and hence they must coincide. This completes the proof. \square

3.2. Relationships between atoms and molecules. In sake of convenience, given a ball B set

$$(3.10) \quad U_0(B) = B, B_i = 2^i B, U_i(B) = 2^i B \setminus 2^{i-1} B, \quad i = 1, 2, \dots$$

Definition 3.8. Let $\epsilon > 0$. A function $m \in L^2(\mathbb{R}^n)$ is called a $(2, M, \lambda, \epsilon)$ -molecule associated to L if there exist a function $b \in \mathcal{D}(L^M)$ and a ball B such that

- $m = L^M b$;
- For every $k = 0, 1, 2, \dots, M$ and $j = 0, 1, 2, \dots$, there holds

$$\|(r_B^2 L)^k b\|_{L^2(U_j(B))} \leq r_B^{2M} 2^{-j\epsilon} (2^j r_B)^{-\lambda/2},$$

where the annuli $U_j(B)$ have been defined in (3.10).

Definition 3.9. Given $M \geq 1, \lambda \in (0, n)$ and $\epsilon > 0$. We say that $\sum \lambda_j m_j$ is a molecular $(2, M, \lambda, \epsilon)$ -representation of f provided that $\{\lambda_j\}_{j=0}^\infty \in \ell^1$, each m_j is a $(2, M, \lambda, \epsilon)$ -molecule, and the sum converges in $L^2(\mathbb{R}^n)$. Set

$$\mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n) = \left\{ f : f \text{ has a molecular } (2, M, \lambda, \epsilon)\text{-representation} \right\},$$

with the norm given by

$$\begin{aligned} & \|f\|_{\mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n)} \\ &= \inf \left\{ \sum_{j=0}^\infty |\lambda_j| : f = \sum_{j=0}^\infty \lambda_j m_j \text{ is a molecular } (2, M, \lambda, \epsilon)\text{-representation} \right\}. \end{aligned}$$

The space $F\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n)$ is then defined as the completion of $\mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n)$ with respect to this norm.

Lemma 3.10. Assume L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying (1.1). Let $\epsilon > 0$. Then

$$\mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n) \subseteq F\dot{H}_L^\lambda(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

and

$$\|f\|_{F\dot{H}_L^\lambda(\mathbb{R}^n)} \leq C \|f\|_{\mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n)}.$$

Proof. By Lemma 2.3 and Definition 3.1, and an application of the L^2 boundedness of Q_{t^2} , we see that it will be enough to show that for every $(2, M, \lambda, \epsilon)$ -molecule m we have $\|Q_{t^2}(m)\|_{F\dot{T}^\lambda} \leq C$. To this end, let $\epsilon > 0$, m be a $(2, M, \lambda, \epsilon)$ -molecule, adapted to the ball $B = B(x_B, r_B)$. Fix $\tilde{\omega}$ as in (3.6) with $\delta < \min((n - \lambda), 2\epsilon)$. Recalling (3.10), we write

$$\begin{aligned} \|Q_{t^2}(m)\|_{F\dot{T}^\lambda} &\leq \left(\int_{\mathbb{R}_+^{n+1}} \frac{|Q_{t^2}(m\chi_{U_i(B)})(x)|^2 dx dt}{\tilde{\omega}(x, t)} \frac{1}{t} \right)^{1/2} \\ &\leq \sum_{i=0}^{\infty} \left(\int_{\mathbb{R}_+^{n+1}} \frac{|Q_{t^2}(m\chi_{U_i(B)})(x)|^2 dx dt}{\tilde{\omega}(x, t)} \frac{1}{t} \right)^{1/2} \\ &=: \sum_{i=0}^{\infty} I_i. \end{aligned}$$

For $i = 0$, an argument similar to that of Theorem 3.7 yields $I_0 \leq C$. And, for $i \geq 1$ one has

$$\begin{aligned} I_i &\leq \sum_{j=2}^{\infty} \left(\int_{T(2^j B_i) \setminus T(2^{j-1} B_i)} \frac{|Q_{t^2}(m\chi_{U_i(B)})(x)|^2 dx dt}{\tilde{\omega}(x, t)} \frac{1}{t} \right)^{1/2} \\ &\quad + \left(\int_{T(2B_i)} \frac{|Q_{t^2}(m\chi_{U_i(B)})(x)|^2 dx dt}{\tilde{\omega}(x, t)} \frac{1}{t} \right)^{1/2} \\ &= \left(\sum_{j=2}^{\infty} I_{ij} \right) + I_{i1}. \end{aligned}$$

To control I_{ij} ($j = 2, 3, \dots$), note that

$$(x, t) \in T(2^{j+i} B) \setminus T(2^{j+i-1} B) \implies \tilde{\omega}(x, t) \geq Cr_B^{-\lambda} 2^{-(j+i)(\lambda+\delta)}.$$

So, a combination of (3.1), the definition of molecules and Hölder's inequality, produces

$$\begin{aligned} I_{ij}^2 &\leq Cr_B^\lambda 2^{(i+j)(\lambda+\delta)} \int_{T(2^j B_i) \setminus T(2^{j-1} B_i)} \frac{|Q_{t^2}(m\chi_{U_i(B)})(x)|^2 dx dt}{t} \\ &\leq Cr_B^\lambda 2^{(i+j)(\lambda+\delta)} \int_{T(2^j B_i) \setminus T(2^{j-1} B_i)} \left(\int \frac{t|m(y)\chi_{U_i(B)}(y)|}{(t + |x - y|)^{n+1}} dy \right)^2 \frac{dx dt}{t} \\ (3.11) \quad &\leq Cr_B^\lambda 2^{(i+j)(\lambda+\delta)} \int_{T(2^j B_i)} \frac{t^2}{(2^{i+j-2} r_B)^{2n+2}} \frac{dx dt}{t} \|m\|_{L^1(U_i(B))}^2 \\ &\leq Cr_B^\lambda 2^{(i+j)(\lambda+\delta)} \frac{(2^{i+j} r_B)^{2+n}}{(2^{i+j} r_B)^{2n+2}} \|m\|_{L^2(U_i(B))}^2 |2^i B| \\ &\leq C 2^{-j(n-\lambda-\delta)} 2^{-i(2\epsilon-\delta)}. \end{aligned}$$

To estimate I_{i1} , note that

$$(x, t) \in T(2B_i) = T(2^{i+1} B) \implies \tilde{\omega}(x, t) \geq Cr_B^{-\lambda} 2^{-i(\lambda+\delta)}.$$

Thus, an application of the definition of molecules and (3.1) yields

$$\begin{aligned}
 I_{i1}^2 &\leq Cr_B^\lambda 2^{i(\lambda+\delta)} \int_{T(B_i)} |Q_{t^2}(m\chi_{U_i(B)})(x)|^2 \frac{dxdt}{t} \\
 (3.12) \quad &\leq Cr_B^\lambda 2^{i(\lambda+\delta)} \|m\|_{L^2(U_i(B))}^2 \\
 &\leq C2^{-i(2\epsilon-\delta)}.
 \end{aligned}$$

Combining (3.11) and (3.12), we get $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} I_{ij} \leq C$, thereby completing the proof. \square

As an immediate consequence, we get the following result.

Theorem 3.11. *Suppose $(M, \lambda, \epsilon) \in \mathbb{N} \times (0, n) \times (0, \infty)$ and L is a nonnegative self-adjoint operator obeying (1.1). There holds*

$$F\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n) = F\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n) = F\dot{H}_L^\lambda(\mathbb{R}^n).$$

Moreover

$$\|f\|_{F\dot{H}_{L,at,M}^\lambda} \approx \|f\|_{F\dot{H}_L^\lambda} \approx \|f\|_{F\dot{H}_{L,mol,M,\epsilon}^\lambda},$$

where the implicit constants depend only on the triple (M, λ, ϵ) and the constant in (1.1).

Proof. We have already shown that

$$F\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n) = F\dot{H}_L^\lambda(\mathbb{R}^n) \quad \& \quad \mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n) = F\dot{H}_L^\lambda(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),$$

with equivalent norms. Moreover, every $(2, M, \lambda)$ -atom is, in particular, a $(2, M, \lambda, \epsilon)$ -molecule for every $\epsilon > 0$, hence

$$\mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n) \subseteq \mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n)$$

with

$$\|f\|_{\mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n)} \leq \|f\|_{\mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)} \quad \forall \quad f \in \mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n).$$

Also, by Lemma 3.10 one has

$$\mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n) \subseteq F\dot{H}_L^\lambda(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = \mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)$$

with

$$\|f\|_{\mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)} \approx \|f\|_{F\dot{H}_L^\lambda(\mathbb{R}^n)} \leq C\|f\|_{\mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n)}.$$

Consequently,

$$\mathbb{F}\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n) = F\dot{H}_L^\lambda(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = \mathbb{F}\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n),$$

with equivalent norms. It follows that the three completions

$$F\dot{H}_L^\lambda(\mathbb{R}^n); \quad F\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n); \quad F\dot{H}_{L,mol,M,\epsilon}^\lambda(\mathbb{R}^n)$$

coincide for different choices of $(M, \lambda, \epsilon) \in \mathbb{N} \times (0, n) \times (0, \infty)$. \square

The following representation will be used later on.

Theorem 3.12. *Assume L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying (1.1). Let $M \geq 1$. Suppose $f = \sum_{i=0}^N \lambda_i a_i$, where $\{a_i\}_{i=0}^N$ is a family of $(2, 2M, \lambda)$ -atoms and $\sum_{i=0}^N |\lambda_i| < \infty$. Then there is a representation of $f = \sum_{i=0}^K \mu_i m_i$, where the m_i 's are $(2, M, \lambda, M)$ -molecules and*

$$C_1 \|f\|_{\dot{\mathbb{F}}_{L,at,M}^\lambda(\mathbb{R}^n)} \leq \sum_{i=0}^K |\mu_i| \leq C_2 \|f\|_{\dot{\mathbb{F}}_{L,at,M}^\lambda(\mathbb{R}^n)},$$

with $C_j = C_j(L, M, \lambda, n)$ for $j = 1, 2$.

Proof. This follows from a slight modification of the argument for [20, Theorem 5.4]. \square

4. IDENTIFICATION BETWEEN $(F\dot{H}_L^\lambda(\mathbb{R}^n))^*$ AND $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$

4.1. A characterization of $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$.

Definition 4.1. *Given a nonnegative self-adjoint operator L on $L^2(\mathbb{R}^n)$ satisfying (1.1). For $(\nu, \lambda, \epsilon) \in \mathcal{D}(L) \times (0, n) \times (0, \infty)$ let*

$$\phi = L\nu \in L^2(\mathbb{R}^n) \quad \& \quad \|\phi\|_{\mathcal{M}^{2,\lambda,\epsilon}(L)} := \sup_{j \geq 0} \left[2^{j\epsilon} 2^{j\frac{\lambda}{2}} \sum_{k=0}^1 \|L^k \nu\|_{L^2(U_j(B_0))} \right],$$

where B_0 is the ball centered at some $x_0 \in \mathbb{R}^n$ with radius 1. Then

$$\mathcal{M}^{2,\lambda,\epsilon}(L) := \{\phi = L\nu \in L^2(\mathbb{R}^n) : \|\phi\|_{\mathcal{M}^{2,\lambda,\epsilon}(L)} < \infty\}.$$

The following two facts are worth mentioning:

- if $\phi \in \mathcal{M}^{2,\lambda,\epsilon}(L)$ with norm 1, then ϕ is a $(2, 1, \lambda, \epsilon)$ -molecule adapted to B_0 . Conversely, if m is a $(2, 1, \lambda, \epsilon)$ -molecule adapted to any ball, then $m \in \mathcal{M}^{2,\lambda,\epsilon}(L)$.
- if $(\mathcal{M}^{2,\lambda,\epsilon}(L))^*$ stands for the dual of $\mathcal{M}^{2,\lambda,\epsilon}(L)$ and A_t denotes either $(I + t^2 L)^{-1}$ or $e^{-t^2 L}$, then $(\mathcal{M}^{2,\lambda,\epsilon}(L))^* \ni f \mapsto (I - A_t)f$ can be determined in the sense of distribution and so this mapping belongs to $L_{\text{loc}}^2(\mathbb{R}^n)$ – indeed, if $\varphi \in L^2(B)$ for some ball B , it follows that $(I - A_t)\varphi \in \mathcal{M}^{2,\lambda,\epsilon}(L)$ for every $\epsilon > 0$, and so that

$$|\langle (I - A_t)f, \varphi \rangle| = |\langle f, (I - A_t)\varphi \rangle| \leq C_{t, r_B \text{ dist}(B, x_0)} \|f\|_{(\mathcal{M}^{2,\lambda,\epsilon}(L))^*} \|\varphi\|_{L^2(B)}.$$

Similarly, one has $(t^2 L)A_t f \in L_{\text{loc}}^2(\mathbb{R}^n)$.

Definition 4.2. *Given $\lambda \in (0, n)$ and a nonnegative self-adjoint operator L on $L^2(\mathbb{R}^n)$ satisfying (1.1). Let*

$$\mathcal{E}_\lambda := \bigcap_{\epsilon > 0} (\mathcal{M}^{2,\lambda,\epsilon}(L))^*.$$

Then an element $f \in \mathcal{E}_\lambda$ is said to belong to $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ provided

$$\|f\|_{\mathcal{L}_L^{2,\lambda}} := \sup_{B \subset \mathbb{R}^n} \left(\frac{1}{r_B^\lambda} \int_B |(I - e^{-r_B^2 L})f(x)|^2 dx \right)^{1/2} < \infty.$$

It is worth remarking that Definition 4.2 is essentially equivalent to the original definition of a quadratic Campanato space associated to L introduced in [16]. With this in mind, the following description of $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ is quite natural.

Lemma 4.3. *Assume L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying (1.1). Let $\lambda \in (0, n)$. An element $f \in \mathcal{E}_\lambda$ belongs to $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ if and only if*

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{r_B^\lambda} \int_B |(I - (I + r_B^2 L)^{-1})f(x)|^2 dx \right)^{1/2} < \infty.$$

Proof. This follows from a minor change of the argument for [21, Lemma 8.1]. \square

4.2. Preduality for $\mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$. By Theorem 3.11, we denote $F\dot{H}_{L,at,M}^\lambda(\mathbb{R}^n)$ by $F\dot{H}_{L,at}^\lambda(\mathbb{R}^n)$. Theorem 1.1 is split into two parts: Theorem 4.4 and its converse Theorem 4.5 below.

Theorem 4.4. *Assume L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying (1.1). Let $\lambda \in (0, n)$ and $M \geq 1$. Then for any $f \in \mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$, the linear functional*

$$\ell(g) := \langle f, g \rangle,$$

which is initially defined on the dense subspace of $\mathcal{M}^{2,\lambda,\epsilon}(L)$ comprising finite linear combinations of $(2, 1, \lambda, \epsilon)$ -molecules, $\epsilon > \frac{n-\lambda}{2}$ and where the pairing \langle, \rangle acts between $\mathcal{M}^{2,\lambda,\epsilon}(L)$ and its dual, has a unique bounded extension to $F\dot{H}_{L,at}^\lambda(\mathbb{R}^n)$ with

$$\|\ell\|_{(F\dot{H}_{L,at}^\lambda)^*} \leq C \|f\|_{\mathcal{L}_L^{2,\lambda}}, \quad \text{for some } C \text{ independent of } f.$$

Proof. Let us prove first that given $(2, 1, \lambda, \epsilon)$ -molecule m with $\epsilon > \frac{n-\lambda}{2}$ one has

$$(4.1) \quad |\langle f, m \rangle| \leq C \|f\|_{\mathcal{L}_L^{2,\lambda}} \quad \forall \quad f \in \mathcal{L}_L^{2,\lambda}(\mathbb{R}^n).$$

Note that if $f \in \mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$ then $f \in (\mathcal{M}^{2,\lambda,\epsilon}(L))^*$ and hence $(I - (I + r_B^2 L)^{-1})f \in L_{loc}^2$. Thus, with B denoting the ball associated with m , we may write

$$\langle f, m \rangle = \int_{\mathbb{R}^n} (I - (I + r_B^2 L)^{-1})f(x) \overline{m(x)} dx + \left\langle (I + r_B^2 L)^{-1}f, m \right\rangle =: I_1 + I_2.$$

So, (4.1) follows from controlling I_1 and I_2 from above.

For the term I_1 , we apply Cauchy–Schwarz’s inequality, the L^2 -normalization of m and (3.10) to obtain

$$\begin{aligned} |I_1| &\leq \sum_{j=0}^{\infty} \left(\int_{U_j(B)} |(I - (I + r_B^2 L)^{-1})f(x)|^2 dx \right)^{1/2} \left(\int_{U_j(B)} |m(x)|^2 dx \right)^{1/2} \\ &\leq \sum_{j=0}^{\infty} 2^{-j\epsilon} (2^j r_B)^{-\lambda/2} \left(\int_{U_j(B)} |(I - (I + r_B^2 L)^{-1})f(x)|^2 dx \right)^{1/2}. \end{aligned}$$

Upon covering $U_j(B)$ by approximate 2^{jn} balls of the radius r_B and using $\epsilon > (n - \lambda)/2$, we obtain

$$|I_1| \leq \sum_{j=0}^{\infty} 2^{-j\epsilon} (2^j r_B)^{-\lambda/2} 2^{jn/2} r_B^{\lambda/2} \|f\|_{\mathcal{L}_L^{2,\lambda}}$$

$$\begin{aligned}
&= \sum_{j=0}^{\infty} 2^{-j\epsilon} 2^{j(n/2-\lambda/2)} \|f\|_{\mathcal{L}_L^{2,\lambda}} \\
&\leq C \|f\|_{\mathcal{L}_L^{2,\lambda}}.
\end{aligned}$$

For the term I_2 , we use

$$L(I + r_B^2 L)^{-1} = r_B^{-2} (I - (I + r_B^2 L)^{-1})$$

and the definition of a $(2, 1, \lambda, \epsilon)$ -molecule to derive

$$\begin{aligned}
|I_2| &\leq C \left| \int_{\mathbb{R}^n} (I - (I + r_B^2 L)^{-1}) f(x) \overline{(r_B^2 L)^{-1} m(x)} dx \right| \\
&\leq C \sum_{j=0}^{\infty} \left(\int_{U_j(B)} |(I - (I + r_B^2 L)^{-1}) f(x)|^2 dx \right)^{1/2} \left(\int_{U_j(B)} |(r_B^2 L)^{-1} m(x)|^2 dx \right)^{1/2} \\
&\leq C \sum_{j=0}^{\infty} 2^{-j\epsilon} (2^j r_B)^{-\lambda/2} \left(\int_{U_j(B)} |(I - (I + r_B^2 L)^{-1}) f(x)|^2 dx \right)^{1/2}.
\end{aligned}$$

Furthermore, upon covering each $U_j(B)$ by approximate 2^{jn} balls of the radius r_B , we obtain

$$|I_2| \leq \sum_{j=0}^{\infty} 2^{-j\epsilon} (2^j r_B)^{-\lambda/2} 2^{jn/2} r_B^{\lambda/2} \|f\|_{\mathcal{L}_L^{2,\lambda}} \leq C \|f\|_{\mathcal{L}_L^{2,\lambda}}.$$

Our next goal is to show that for every $N \in \mathbb{N}$ and for every $g = \sum_{j=0}^N \lambda_j a_j \in F\dot{H}_{L,at}^\lambda(\mathbb{R}^n)$, where $\{a_j\}_{j=0}^N$ are $(2, 2n, \lambda)$ -atoms, we have

$$(4.2) \quad \left| \int_{\mathbb{R}^n} f(x) g(x) dx \right| \leq C \|g\|_{F\dot{H}_{L,at}^\lambda} \|f\|_{\mathcal{L}_L^{2,\lambda}}.$$

Since the space of all finite linear combinations of $(2, 2n, \lambda)$ -atoms is dense in $F\dot{H}_{L,at}^\lambda$, the linear functional ℓ will then have a unique bounded extension to $F\dot{H}_{L,at}^\lambda$ defined in a standard fashion by continuity. Below is a demonstration of (4.2). By Theorem 3.12, there is a representation of

$$g = \sum_{j=0}^N \lambda_j a_j = \sum_{i=0}^K \mu_i m_i,$$

where $\{m_i\}_{i=0}^K$ are $(2, n, \lambda, n)$ -molecules (of course, they are $(2, 1, \lambda, n)$ -molecules) and

$$\sum_{i=0}^K |\mu_i| \leq C \|g\|_{F\dot{H}_{L,at}^\lambda}.$$

Therefore, by (4.1) we have

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} f(x) g(x) dx \right| &\leq \sum_{i=0}^K |\mu_i| \left| \int_{\mathbb{R}^n} f(x) m_i(x) dx \right| \\
&\leq C \sum_{i=0}^K |\mu_i| \|f\|_{\mathcal{L}_L^{2,\lambda}}
\end{aligned}$$

$$\leq C \|g\|_{F\dot{H}_{L,at}^\lambda} \|f\|_{\mathcal{L}_L^{2,\lambda}},$$

whence reaching (4.2) which in turn finishes the proof of Theorem 4.4. \square

Our next result is essentially the converse of Theorem 4.4.

Theorem 4.5. *Assume L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying (1.1). Let $\epsilon > 0$ and $\lambda \in (0, n)$. Suppose ℓ is a bounded linear functional on $F\dot{H}_{L,at}^\lambda(\mathbb{R}^n)$. Then $\ell \in \mathcal{L}_L^{2,\lambda}(\mathbb{R}^n)$, and for any $g \in F\dot{H}_{L,at}^\lambda(\mathbb{R}^n)$ (which can be represented as finite linear combinations of $(2, 1, \lambda, \epsilon)$ -molecules) there holds*

$$(4.3) \quad \ell(g) = \langle \ell, g \rangle,$$

where the pairing is that between $\mathcal{M}^{2,\lambda,\epsilon}(L)$ and its dual. Moreover,

$$\|\ell\|_{\mathcal{L}_L^{2,\lambda}} \leq C \|\ell\|_{(F\dot{H}_{L,at}^\lambda)^*}.$$

Proof. By Theorem 3.10, we have that for any $(2, 1, \lambda, \epsilon)$ -molecule m ,

$$\|m\|_{F\dot{H}_{L,at}^\lambda} \leq C \quad \text{and so} \quad |\ell(m)| \leq C \|\ell\|_{(F\dot{H}_{L,at}^\lambda)^*}.$$

By the discussion on Definition 4.1, ℓ is a bounded linear functional on $\mathcal{M}^{2,\lambda,\epsilon}(L)$ for any $\epsilon > 0$. Thus, $\ell \in \mathcal{E}_\lambda$ and (4.3) holds. Further, $(I - (I + r_B^2 L)^{-1})\ell$ is well defined and belongs to $L_{\text{loc}}^2(\mathbb{R}^n)$. Fix a ball B , and let $\varphi \in L^2(B)$, with $\|\varphi\|_{L^2(B)} \leq 1$. As we observed before,

$$\tilde{m} := r_B^{-\lambda/2} (I - (I + r_B^2 L)^{-1})\varphi$$

is (up to a multiplicative constant) a $(2, 1, \lambda, \epsilon)$ -molecule. Thus,

$$\begin{aligned} r_B^{-\lambda/2} |\langle (I - (I + r_B^2 L)^{-1})\ell, \varphi \rangle| &= r_B^{-\lambda/2} |\langle \ell, (I - (I + r_B^2 L)^{-1})\varphi \rangle| \\ &= |\langle \ell, \tilde{m} \rangle| \\ &\leq C \|\ell\|_{(F\dot{H}_{L,at}^\lambda)^*}. \end{aligned}$$

Taking the supremum over all such φ supported in B , we obtain

$$\frac{1}{r_B^\lambda} \int_B |(I - (I + r_B^2 L)^{-1})\ell(x)|^2 dx \leq C \|\ell\|_{(F\dot{H}_{L,at}^\lambda)^*}^2.$$

Finally, taking the supremum over all balls B in \mathbb{R}^n , we arrive at the conclusion of Theorem 4.5. \square

Proof of Theorem 1.1. Combining Theorem 4.4, Theorem 4.5 and Theorem 3.11, we get

$$(F\dot{H}_L^\lambda(\mathbb{R}^n))^* = (F\dot{H}_{L,at}^\lambda(\mathbb{R}^n))^* = \mathcal{L}_L^{2,\lambda}(\mathbb{R}^n),$$

thereby reaching the preduality stated in Theorem 1.1. \square

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